REGULARITY OF MINIMIZERS OF SEMILINEAR ELLIPTIC PROBLEMS UP TO DIMENSION FOUR

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Abstract. We consider the class of semi-stable solutions to semilinear equations $-\Delta u = f(u)$ in a bounded smooth domain Ω of $\mathbb{R}^n$ (with Ω convex in some results). This class includes all local minimizers, minimal, and extremal solutions. In dimensions $n \leq 4$, we establish an priori L^∞ bound which holds for every positive semi-stable solution and every nonlinearity f. This estimate leads to the boundedness of all extremal solutions when $n = 4$ and Ω is convex. This result was previously known only in dimensions $n \leq 3$ by a result of G. Nedev. In dimensions $5 \leq n \leq 9$ the boundedness of all extremal solutions remains an open question. It is only known to hold in the radial case $\Omega = B_R$ by a result of A. Capella and the author.

1. Introduction and results

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^∞ function and F a primitive of f, i.e. $F' = f$. Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞ domain. Consider the energy functional

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u)\, dx. \tag{1.1}$$

Its Euler-Lagrange equation, under zero boundary conditions, is given by

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We say that a function $u \in C^1_0(\overline{\Omega})$ (i.e., a $C^1(\overline{\Omega})$ function vanishing on $\partial\Omega$) is a *local minimizer* of (1.1) if there exists $\varepsilon > 0$ such that

$$E(u) \leq E(u + \xi)$$

for every $C^1_0(\overline{\Omega})$ function ξ with $\|\xi\|_{C^1(\overline{\Omega})} \leq \varepsilon$. By elliptic regularity, every local minimizer u is a C^∞ classical solution of (1.2). In addition, it is a *semi-stable* solution in the following sense.

We say that a classical solution $u \in C^2(\overline{\Omega})$ of (1.2) is *semi-stable* if

$$Q_u(\xi) := \int_\Omega |\nabla \xi|^2 - f'(u)\xi^2\, dx \geq 0 \tag{1.3}$$

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for every $\xi \in C_0^1(\overline{\Omega})$. Note that Q_u is the second variation of energy at u. The semi-stability of a solution u is equivalent to the condition $\lambda_1 \geq 0$, where $\lambda_1 = \lambda_1(-\Delta - f'(u); \Omega)$ is the first Dirichlet eigenvalue of the linearized operator $-\Delta - f'(u)$ at u in Ω. We use the name *semi-stable* to distinguish from the notion of *stable* solution, defined by $\lambda_1(-\Delta - f'(u); \Omega) > 0$. Note that a local minimizer is always semi-stable, but not necessarily a stable solution.

The following is our main estimate. It originates from questions raised by H. Brezis during the nineties (see [2]) on certain extremal solutions described below. In dimensions $n \leq 4$, we bound the $L^\infty(\Omega)$ norm of every positive semi-stable solution u by the $W^{1,4}$ norm of u on the set $\{u < t\}$ —where t can be chosen arbitrarily. The estimate holds in every smooth domain Ω (not necessarily convex) and for every nonlinearity f —the estimate is indeed completely independent of f. The importance of the bound is that, by choosing t small, $\{u < t\}$ becomes a small neighborhood of $\partial\Omega$, and thus the boundedness of u in Ω is reduced to a question on the regularity of u near $\partial\Omega$.

Theorem 1.1. *Let f be any C^∞ function and $\Omega \subset \mathbb{R}^n$ any C^∞ bounded domain. Assume that $2 \leq n \leq 4$.*

Let $u \in C_0^1(\overline{\Omega})$, with $u > 0$ in Ω, be a local minimizer of (1.1)*, or more generally a classical semi-stable solution of* (1.2)*. Then, for every $t > 0$,*

$$\|u\|_{L^\infty(\Omega)} \leq t + \frac{C}{t}|\Omega|^{(4-n)/(2n)} \left(\int_{\{u<t\}} |\nabla u|^4 \, dx \right)^{1/2}, \tag{1.4}$$

where C is a universal constant (*in particular, independent of f, Ω, and u*)*. In the last integral we have used the notation $\{u < t\} = \{x \in \Omega \, : \, u(x) < t\}$.*

We will be able to control the right hand side of (1.4), for t small enough, if Ω is convex. The reason is that in convex domains, the moving planes method leads to boundary estimates for all positive solutions and all nonlinearities f.

Note that (1.4) is invariant under dilations of the domain Ω, and also allows to multiply u by a constant (by chosing t to have the same units as u).

The following is the main application of Theorem 1.1. It motivated this work. Consider the problem

$$\left\{ \begin{array}{rcll} -\Delta u &=& \lambda g(u) & \text{in } \Omega \\ u &\geq& 0 & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1.5_λ}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $n \geq 2$, $\lambda \geq 0$, and the nonlinearity $g : [0, +\infty) \to \mathbb{R}$ satisfies

$$g \text{ is } C^1, \text{nondecreasing, } g(0) > 0, \text{ and } \lim_{u \to +\infty} \frac{g(u)}{u} = +\infty. \tag{1.6}$$

It is well known (see [2, 4, 6] and references therein) that there exists an extremal parameter $\lambda^* \in (0, \infty)$ such that if $0 \leq \lambda < \lambda^*$ then (1.5_λ) admits a

minimal classical solution u_λ. In addition, this solution u_λ is semi-stable —see Remark 1.3 below. On the other hand, if $\lambda > \lambda^*$ then (1.5_λ) has no classical solution. Here, classical means bounded, while minimal means smallest. The set $\{u_\lambda : 0 \le \lambda < \lambda^*\}$ is increasing in λ and its limit as $\lambda \nearrow \lambda^*$ is a weak solution $u^* = u_{\lambda^*}$ of (1.5_{λ^*}), called the extremal solution of (1.5_λ). Later we will give the precise meaning of weak solution.

When $g(u) = e^u$, it is known that $u^* \in L^\infty(\Omega)$ if $n \le 9$ (for every Ω), while $u^*(x) = -2\log|x|$ if $n \ge 10$ and $\Omega = B_1$. A similar phenomenon happens when $g(u) = (1+u)^p$ with $p > 1$. These results date from the seventies. In the nineties important progress in the subject came from works of H. Brezis and collaborators. He raised the question (see Brezis [2] and also Brezis and Vázquez [4]) of determining the regularity of u^*, depending on the dimension n, for general convex nonlinearities g satisfying (1.6). The best known result was established in 2000 by G. Nedev [21]. He proved that, for every domain Ω and convex nonlinearity g satisfying (1.6), $u^* \in L^\infty(\Omega)$ if $n \le 3$, while $u^* \in H_0^1(\Omega)$ if $n \le 5$.

In this article, using Theorem 1.1 we establish the boundedness of u^* in dimensions $n \le 4$ when Ω is a convex domain. We only assume (1.6) on the nonlinearity g. In dimension 2, the convexity of Ω is not assumed.

Theorem 1.2. *Let g satisfy* (1.6) *and $\Omega \subset \mathbb{R}^n$ be a C^∞ bounded domain. Assume that $2 \le n \le 4$, and that Ω is convex in case $n \in \{3, 4\}$.*

Let u^* be the extremal solution of (1.5_λ). *Then, $u^* \in L^\infty(\Omega)$.*

The validity of this result in nonconvex domains is an open question.

The boundedness of u^* in dimensions $5 \le n \le 9$ remains an open problem. However, in the radial case $\Omega = B_1$, Capella and the author [7] have established that $u^* \in L^\infty(B_1)$ whenever $n \le 9$, for all nonlinearities g satisfying (1.6). See also [7] for precise pointwise bounds for u^* in the radial case, and Villegas [25] for improvements of some of them. These pointwise bounds hold not only for extremal solutions u^* but for all semi-stable solutions.

For $n \in \{2, 3\}$ and general Ω, Nedev [21] assumed g to be convex (a classical hypothesis in the literature of the subject) and to satisfy (1.6). Our result holds also for nonconvex g, but when $n = 3$ we need to assume Ω to be convex.

In [10], Sanchón and the author use some ideas of the present paper to establish new $L^q(\Omega)$ estimates for semi-stable and extremal solutions of $-\Delta u = f(u)$ in dimensions $n \ge 5$ for general nonlinearities f, as well as related bounds for semilinear equations involving the fractions of the Laplacian.

The articles [9, 13, 22] contain extensions of the previous results on extremal solutions for more general quasilinear problems and problems related to nonlinear Neumann boundary conditions.

Theorem 1.2 on the boundedness of u^* will follow easily from our main estimate (1.4) of Theorem 1.1. The key point is that the minimal solutions u_λ of (1.5_λ) are all semi-stable (see next remark). The boundary estimate needed

to control the right hand side of (1.4) is known to hold in convex domains —it follows from the moving planes method. Thus, we only assume the convexity of Ω to guarantee boundary estimates for the solution.

Remark 1.3. That the minimal solutions u_λ of (1.5_λ) are semi-stable can be easily seen as follows. Since there is no solution smaller than u_λ, we have that u_λ must coincide with the absolute minimizer of the energy in the close convex set of functions lying between 0 (a strict subsolution) and u_λ, a (super) solution. That is, u_λ is a one-sided minimizer (by below). Now, consider small perturbations $u_\lambda+\varepsilon\xi$, with $\xi \le 0$, lying in this convex set. Since u_λ is a critical point of the energy and minimizes it for these perturbations, we deduce that the second variation $Q_{u_\lambda}(\xi) \ge 0$ for all $\xi \in C_0^1\left(\overline{\Omega}\right)$ with $\xi \le 0$ in Ω. Now, writting any ξ as $\xi = \xi^+ - \xi^-$ (its positive and negative parts) and using the form of $Q_{u_\lambda}(\xi)$ and that ξ^+ and ξ^- have disjoint supports, we conclude $Q_{u_\lambda}(\xi) \ge 0$ for all $\xi \in C_0^1\left(\overline{\Omega}\right)$, as claimed.

Let us mention that the solutions $\{u_\lambda : 0 \le \lambda \le \lambda^*\}$ can also be obtained using the implicit function theorem (starting from $\lambda = 0$) when g is convex (see [4]). This gives an alternative way of proving the semi-stability of u_λ. Another possibility is to obtain u_λ by the monotone iteration method, with $0 < \lambda < \lambda^*$ fixed, starting from the strict subsolution $u \equiv 0$. This is the reason why u_λ is the minimal (or smallest) solution. In addition, its semi-stability can also be proved via this monotone iteration procedure.

See [12], and references therein, for more information on minimal and extremal solutions, also for more general operators.

The following result states a more explicit $L^\infty(\Omega)$ estimate for semi-stable solutions. The estimate depends on certain assumptions on the nonlinearity which are more general than (1.6). It applies to certain weak solutions. We recall that $u \in L^1(\Omega)$ is said to be a weak solution of (1.2) if $f(u)\mathrm{dist}(\cdot, \partial\Omega) \in L^1(\Omega)$ and $\int_\Omega u\Delta\xi + f(u)\xi \, dx = 0$ for all $\xi \in C^2(\overline{\Omega})$ with $\xi \equiv 0$ on $\partial\Omega$. Theorem 1.2 on extremal solutions will be an easy consequence of the next result.

Theorem 1.4. *Let f be any C^∞ function and $\Omega \subset \mathbb{R}^n$ any C^∞ bounded domain. Assume that $2 \le n \le 4$, and that Ω is convex in case $n \in \{3, 4\}$.*

Let $u \in L^1(\Omega)$ be a positive weak solution of (1.2) *and suppose that u is the $L^1(\Omega)$ limit of a sequence of classical positive semi-stable solutions of* (1.2). *We then have:*

(i) *If $f \ge 0$ in $[0, \infty)$, then $u \in L^\infty(\Omega)$.*

(ii) *Assume that*

$$f(s) \ge c_1 > 0 \text{ and } f(s) \ge \mu s - c_2 \quad \text{for all } s \in [0, \infty), \tag{1.7}$$

for some positive constants c_1 *and* c_2 *and for some* $\mu > \lambda_1(\Omega)$*, where* $\lambda_1(\Omega)$ *is the first Dirichlet eigenvalue of* $-\Delta$ *in* Ω*. Then,*

$$\|u\|_{L^\infty(\Omega)} \leq C\left(\Omega, \mu, c_1, c_2, \|f\|_{L^\infty(\,[0,\overline{C}(\Omega,\mu,c_2)]\,)}\right), \tag{1.8}$$

where $C(\cdot)$ *and* $\overline{C}(\cdot)$ *are constants depending only on the quantities within the parentheses.*

The proof of our main estimate (1.4) starts by writting the semi-stability condition (1.3) of the solution with the test function ξ replaced by $\xi = c\eta$, where $\eta \in C_0^1\left(\overline{\Omega}\right)$ is still arbitrary and c is a well chosen function satisfying a certain equation for the linearized operator $-\Delta - f'(u)$. This idea was already used to study minimal surfaces (more precisely, minimal cones), where one takes $c = |A|$ (the norm of the second fundamental form of the cone). This motivated our study of the radial semilinear case [7] in which we took $c = \partial_r u$, the radial derivative of u. See [8] for more comments on minimal cones and also on similar ideas for harmonic maps.

In the present paper we take $c = |\nabla u|$, which satisfies

$$(\Delta + f'(u))\,|\nabla u| = \frac{1}{|\nabla u|}\left(|\nabla_T\,|\nabla u||^2 + |A|^2\,|\nabla u|^2\right) \text{ in } \Omega \cap \{|\nabla u| > 0\}\,. \tag{1.9}$$

Here $|A|^2 = |A\,(x)|^2$ is the squared norm of the second fundamental form of the level set of u passing through a given $x \in \Omega \cap \{|\nabla u| > 0\}$, i.e., the sum of the squares of the principal curvatures of such level set. On the other hand, ∇_T denotes the tangential gradient to the level set. Thus, (1.9) contains geometrical information of the level sets of u. This geometrical quantities will appear in the expression for the second variation of energy $Q_u\,(\xi)$ once we take $\xi = |\nabla u|\,\eta$ and, at the same time, the presence of $f'\,(u)$ in (1.3) will disappear since the left hand side of (1.9) refers to $\Delta + f'\,(u)$.

Next, we will take $\eta = \varphi\,(u)$, i.e.,

$$\xi = |\nabla u|\,\varphi\,(u)\,,$$

and we will be lead to a Hardy type inequality (for functions φ of one real variable) involving two weights containing geometrical information of the level sets of u —see inequality (2.8).

The next crucial point in dimension $n = 4$ will be the use of a remarkable Sobolev inequality on general hypersurfaces of $\mathbb{R}^n$ (in our case the level sets of u), which involves the mean curvature of the hypersurface but that has a best constant independent of the hypersurface —in fact, depending only on the dimension n. This Sobolev inequality (Theorem 2.1 below) is due to Michel-Simon [20] and Allard [1].

As we mention in next section, the use of (1.9) and of $\xi = |\nabla u|\,\eta\,(u)$ in the semi-stability condition (1.3) was first exploited by Sternberg and Zumbrun [23, 24] to study semilinear phase transitions problems. Farina [15], and later

Farina-Sciunzi-Valdinoci [16] for more general quasi-linear operators, have also used this method to establish some Liouville type results.

The results of this paper were first announced in March 2006 in a seminar I gave at "Analyse Non Linéaire" (Laboratoire J.L. Lions, Paris VI).

In next section we prove our main estimate, Theorem 1.1. In section 3 we establish Theorem 1.4 and, as a simple consequence, Theorem 1.2.

2. Proof of the main estimate

In this section we prove our main estimate —estimate (1.4) of Theorem 1.1. For this, we will use the following remarkable result.

It is a Sobolev inequality due to Michael and Simon [20] and Allard [1]. It holds on every compact hypersurface of $\mathbb{R}^{m+1}$ without boundary, and its constant is independent of the geometry of the hypersurface.

Theorem 2.1 (**Michael-Simon** [20], **Allard** [1]). *Let $M \subset \mathbb{R}^{m+1}$ be a C^∞ immersed m-dimensional compact hypersurface without boundary.*

Then, for every $p \in [1, m)$, there exists a constant $C = C(m, p)$ depending only on the dimension m and the exponent p such that, for every C^∞ function $v : M \to \mathbb{R}$,

$$\left(\int_M |v|^{p^*} dV\right)^{1/p^*} \leq C(m,p) \left(\int_M |\nabla v|^p + |Hv|^p \ dV\right)^{1/p}, \qquad (2.1)$$

where H is the mean curvature of M and $p^* = mp/(m-p)$.

This inequality is stated in Proposition 5.2 of [19], where references for it and related results are mentioned. In [5] (section 28.5.2) it is stated and proved for $p = 1$.

The geometric Sobolev inequality (2.1) has been used in the PDE literature to obtain estimates for the extinction time of some geometric evolution flows —see, for instance, section F.2 of [14] and also [19].

In the proof of our main estimate in Theorem 1.1 we will use (2.1) with $M = \{u = s\}$ (a level set of u), $v = |\nabla u|^{1/2}$, and $p = 2$.

The level sets of a solution u, and their curvature, appear in the following result of Sternberg and Zumbrun [23, 24]. Its statement is an inequality which follows from the semi-stability hypothesis (1.3) on the solution.

Proposition 2.2 (**Sternberg-Zumbrun** [23, 24]). *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and u a smooth positive semi-stable solution of* (1.2). *Then, for every Lipschitz function η in $\overline{\Omega}$ with $\eta|_{\partial\Omega} \equiv 0$,*

$$\int_{\Omega \cap \{|\nabla u| > 0\}} \left(|\nabla_T |\nabla u||^2 + |A|^2 |\nabla u|^2 \right) \eta^2 \ dx \leq \int_\Omega |\nabla u|^2 |\nabla \eta|^2 \ dx, \qquad (2.2)$$

where ∇_T denotes the tangential or Riemannian gradient along a level set of u (it is thus the orthogonal projection of the full gradient in $\mathbb{R}^n$ along a level

set of u) *and where*

$$|A|^2 = |A(x)|^2 = \sum_{l=1}^{n-1} \kappa_l^2,$$

being κ_l *the principal curvatures of the level set of* u *passing through* x*, for a given* $x \in \Omega \cap \{|\nabla u| > 0\}$.

This result (stated for a Neumann problem instead of a Dirichlet problem) is Lemma 2.1 of [23] and Theorem 4.1 of [24]. The authors conceived and used the result to study qualitative properties of phase transitions in Allen-Cahn equations. For the sake of completeness, we give an elementary proof of it here. See Theorem 2.5 of [16] for a quasilinear extension.

Proof of Proposition 2.2. The semi-stability condition (1.3) also holds, by approximation, for every Lipschitz function ξ in $\overline{\Omega}$ with $\xi|_{\partial\Omega} \equiv 0$. Now, take $\xi = c\eta$ in (1.3), where c is a smooth function and η Lipschitz in $\overline{\Omega}$ and $\eta|_{\partial\Omega} \equiv 0$. A simple integration by parts gives that

$$Q_u(c\eta) = \int_\Omega c^2 \left|\nabla\eta\right|^2 - (\Delta c + f'(u)c)\, c\eta^2 \; dx \geq 0. \tag{2.3}$$

In contrast with [23, 24] (where they took $c = |\nabla u|$) and to avoid some considerations on the set $\{|\nabla u| = 0\}$, we take

$$c = \sqrt{\left|\nabla u\right|^2 + \varepsilon^2}$$

for a given $\varepsilon > 0$. Note that c is smooth.

Since $\Delta u + f(u) = 0$ in Ω, we have $\Delta u_j + f'(u)u_j = 0$ in Ω. We use the notation $u_j = \partial_{x_j} u$ and also $u_{ij} = \partial_{x_i x_j} u$. Using these equations, it is simple to verify that

$$\begin{aligned}\Delta c = \frac{1}{\left|\nabla u\right|^2 + \varepsilon^2} \Bigg\{ & -f'(u) \left|\nabla u\right|^2 \sqrt{\left|\nabla u\right|^2 + \varepsilon^2} \\ & + \sum_{i,j} u_{ij}^2 \sqrt{\left|\nabla u\right|^2 + \varepsilon^2} - \left(\sum_i \left(\sum_j u_{ij} u_j \right)^2 \right) \frac{1}{\sqrt{\left|\nabla u\right|^2 + \varepsilon^2}} \Bigg\},\end{aligned}$$

and thus

$$\begin{aligned}(\Delta + f'(u))\, c = {} & f'(u) \frac{\varepsilon^2}{\sqrt{\left|\nabla u\right|^2 + \varepsilon^2}} \\ & + \frac{1}{\sqrt{\left|\nabla u\right|^2 + \varepsilon^2}} \left\{ \sum_{i,j} u_{ij}^2 - \sum_i \left(\sum_j u_{ij} \frac{u_j}{\sqrt{\left|\nabla u\right|^2 + \varepsilon^2}} \right)^2 \right\}.\end{aligned}$$

Using this equality in (2.3), we deduce

$$\int_\Omega \left(|\nabla u|^2+\varepsilon^2\right)|\nabla\eta|^2\,dx = \int_\Omega c^2\,|\nabla\eta|^2\,dx \tag{2.4}$$

$$\geq \int_\Omega (\Delta c + f'(u)c)\,c\eta^2 dx$$

$$= \int_\Omega f'(u)\varepsilon^2\eta^2 dx \tag{2.5}$$

$$+\int_\Omega \left\{\sum_{i,j} u_{ij}^2 - \sum_i \left(\sum_j u_{ij}\frac{u_j}{\sqrt{|\nabla u|^2+\varepsilon^2}}\right)^2\right\}\eta^2 dx. \tag{2.6}$$

The integrand in the last integral is nonnegative. Thus, we have

$$\int_\Omega \left\{\sum_{i,j} u_{ij}^2 - \sum_i \left(\sum_j u_{ij}\frac{u_j}{\sqrt{|\nabla u|^2+\varepsilon^2}}\right)^2\right\}\eta^2 dx$$

$$\geq \int_{\Omega\cap\{|\nabla u|>0\}} \left\{\sum_{i,j} u_{ij}^2 - \sum_i \left(\sum_j u_{ij}\frac{u_j}{\sqrt{|\nabla u|^2+\varepsilon^2}}\right)^2\right\}\eta^2 dx$$

$$\geq \int_{\Omega\cap\{|\nabla u|>0\}} \left\{\sum_{i,j} u_{ij}^2 - \sum_i \left(\sum_j u_{ij}\frac{u_j}{|\nabla u|}\right)^2\right\}\eta^2 dx.$$

From this and (2.4), (2.5), (2.6), we arrive at

$$\int_\Omega \left(|\nabla u|^2+\varepsilon^2\right)|\nabla\eta|^2\,dx$$

$$\geq \int_\Omega f'(u)\varepsilon^2\eta^2 dx$$

$$+\int_{\Omega\cap\{|\nabla u|>0\}} \left\{\sum_{i,j} u_{ij}^2 - \sum_i \left(\sum_j u_{ij}\frac{u_j}{|\nabla u|}\right)^2\right\}\eta^2 dx.$$

We now let $\varepsilon\downarrow 0$ to obtain

$$\int_\Omega |\nabla u|^2\,|\nabla\eta|^2\,dx \geq \int_{\Omega\cap\{|\nabla u|>0\}} \left\{\sum_{i,j} u_{ij}^2 - \sum_i \left(\sum_j u_{ij}\frac{u_j}{|\nabla u|}\right)^2\right\}\eta^2 dx.$$

We conclude the claimed inequality (2.2) of the proposition since

$$\sum_{i,j} u_{ij}^2 - \sum_i \left(\sum_j u_{ij}\frac{u_j}{|\nabla u|}\right)^2 = |\nabla_T\,|\nabla u||^2 + |A|^2\,|\nabla u|^2 \tag{2.7}$$

at every point $x \in \Omega \cap \{|\nabla u| > 0\}$. This last equality can be easily checked assuming that $\nabla u(x) = (0, \ldots, 0, u_{x_n}(x))$ and looking at the quantities in (2.7) in the orthonormal basis $\{e_1, \ldots, e_{n-1}, (0, \ldots, 0, 1)\}$, where $\{e_1, \ldots, e_{n-1}\}$ are the principal directions of the level set of u through x. See also Lemma 2.1 of [23] for a detailed proof of (2.7). □

Using Proposition 2.2 and Theorem 2.1 we can now establish Theorem 1.1.

Proof of Theorem 1.1. By elliptic regularity, the solution u is smooth, that is, $u \in C^\infty(\overline{\Omega})$. Recall that $u > 0$ in Ω. Let us denote

$$T := \max_{\Omega} u = \|u\|_{L^\infty(\Omega)}$$

and, for $s \in (0, T)$,

$$\Gamma_s := \{x \in \Omega : u(x) = s\}.$$

By Sard's theorem, almost every $s \in (0, T)$ is a regular value of u. By definition, if s is a regular value of u, then $|\nabla u(x)| > 0$ for all $x \in \Omega$ such that $u(x) = s$ (i.e., for all $x \in \Gamma_s$). In particular, if s is a regular value, Γ_s is a C^∞ immersed compact hypersurface of $\mathbb{R}^n$ without boundary. Later we will apply Theorem 2.1 with $M = \Gamma_s$. Note here that, since Γ_s could have a finite number of connected components, inequality (2.1) for connected manifolds M leads to the same inequality (with same constant) for M with more than one component.

Since u is a semi-stable solution, we can use Proposition 2.2. In (2.2) we take

$$\eta(x) = \varphi(u(x)) \quad \text{for} \quad x \in \Omega,$$

where φ is a Lipschitz function in $[0, T]$ with

$$\varphi(0) = 0.$$

The right hand side of (2.2) becomes

$$\begin{aligned} \int_\Omega |\nabla u|^2 |\nabla \eta|^2 \, dx &= \int_\Omega |\nabla u|^4 \varphi'(u)^2 dx \\ &= \int_0^T \left(\int_{\Gamma_s} |\nabla u|^3 \, dV_s \right) \varphi'(s)^2 ds, \end{aligned}$$

by the coarea formula. We have denoted by dV_s the volume element in Γ_s. The integral in ds is over the regular values of u, whose complement is of zero measure in $(0, T)$.

In the left hand side of (2.2) we integrate only on $\Omega \cap \{|\nabla u| > \delta\}$ for a given $\delta > 0$, and thus inequality (2.2) remains valid. Since in this set $|\nabla u|$ is bounded away from zero, the coarea formula gives

$$\begin{aligned}
&\int_0^T \left(\int_{\Gamma_s} |\nabla u|^3 \, dV_s\right) \varphi'(s)^2 ds \\
&\qquad \geq \int_{\Omega\cap\{|\nabla u|>\delta\}} \left(|\nabla_T |\nabla u||^2 + |A|^2 |\nabla u|^2\right) \varphi(u)^2 dx \\
&\qquad = \int_0^T \left(\int_{\Gamma_s\cap\{|\nabla u|>\delta\}} \frac{1}{|\nabla u|} \left(|\nabla_T |\nabla u||^2 + |A|^2 |\nabla u|^2\right) dV_s\right) \varphi(s)^2 ds \\
&\qquad = \int_0^T \left(\int_{\Gamma_s\cap\{|\nabla u|>\delta\}} 4 \left|\nabla_T |\nabla u|^{1/2}\right|^2 + \left(|A| \, |\nabla u|^{1/2}\right)^2 dV_s\right) \varphi(s)^2 ds.
\end{aligned}$$

Letting $\delta \downarrow 0$ and using the monotone convergence theorem, we deduce that

$$\int_0^T h_1(s)\varphi(s)^2 \, ds \leq \int_0^T h_2(s)\varphi'(s)^2 \, ds, \tag{2.8}$$

for all Lipschitz functions $\varphi : [0, T] \to \mathbb{R}$ with $\varphi(0) = 0$, where

$$h_1(s) := \int_{\Gamma_s} 4|\nabla_T|\nabla u|^{1/2}|^2 + \left(|A||\nabla u|^{1/2}\right)^2 \, dV_s \tag{2.9}$$

and

$$h_2(s) := \int_{\Gamma_s} |\nabla u|^3 \, dV_s \tag{2.10}$$

for every regular value s of u.

Inequality (2.8), with h_1 and h_2 as defined above, will lead to our L^∞ estimate of Theorem 1.1 after choosing an appropriate test function φ in (2.8). In dimensions 2 and 3 we will simply use (2.8) and well known geometric inequalities about the curvature of manifolds (note that h_1 involves the curvature of the level sets of u). Instead, in dimension 4 we need the following additional tool. For $n \geq 4$ we use the Michael-Simon and Allard Sobolev inequality (2.1) with $M = \Gamma_s$, $p = 2 < m = n - 1$, and $v = |\nabla u|^{1/2}$. Note that the mean curvature H of Γ_s satisfies $|H| \leq |A|$. We obtain

$$\left(\int_{\Gamma_s} |\nabla u|^{\frac{n-1}{n-3}} \, dV_s\right)^{\frac{n-3}{n-1}} \leq C(n) h_1(s) \tag{2.11}$$

for all regular values s of u, where $C(n)$ us a constant depending only on n. This estimate combined with (2.8) leads to

$$\int_0^T \left(\int_{\Gamma_s} |\nabla u|^{\frac{n-1}{n-3}} \, dV_s\right)^{\frac{n-3}{n-1}} \varphi(s)^2 ds \leq C(n) \int_0^T \left(\int_{\Gamma_s} |\nabla u|^3 \, dV_s\right) \varphi'(s)^2 ds \tag{2.12}$$

for all Lipschitz functions φ in $[0, T]$ with $\varphi(0) = 0$. We only know how to derive an L^∞ estimate for u (i.e., a bound on $T = \max_\Omega u$) from (2.12) when the exponent $(n-1)/(n-3)$ on its left hand side is larger or equal than the

one on the right hand side, i.e., 3. That is, we need $(n-1)/(n-3) \geq 3$, which means $n \leq 4$.

The rest of the proof differs in every dimension $n = 4, 3$, and 2. But in all three cases it will be useful to denote

$$B_t := \frac{1}{t^2} \int_{\{u<t\}} |\nabla u|^4 \, dx = \frac{1}{t^2} \int_0^t h_2(s) ds, \tag{2.13}$$

where $t > 0$ is a given positive constant as in the statement of the theorem. Note that the quantity B_t is the main part of the right hand side of our estimate (1.4). Let us start with the

<u>*Case* $n = 4$</u>. It will be crucial to use the bound obtained in (2.11). It reads, since $(n-1)/(n-3) = 3$,

$$h_2^{1/3} \leq C h_1 \qquad \text{a.e. in } (0,T) \text{ (when } n = 4), \tag{2.14}$$

where C is a universal constant. For every regular value s of u, we have $0 < h_2(s)$ and $h_1(s) < \infty$ (simply by their definition). This together with (2.14) gives $h_1/h_2 \in (0, +\infty)$ a.e. in $(0,T)$. Thus, defining

$$g_k(s) := \min\left\{k, \frac{h_1(s)}{h_2(s)}\right\}$$

for regular values s and for a positive integer k, we have that $g_k \in L^\infty(0,T)$ and

$$g_k(s) \nearrow \frac{h_1(s)}{h_2(s)} \in (0, +\infty) \qquad \text{as } k \uparrow \infty, \text{ for a.e. } s \in (0,T). \tag{2.15}$$

Since $g_k \in L^\infty(0,T)$, the function

$$\varphi_k(s) := \begin{cases} s/t & \text{if} \quad s \leq t, \\ \exp\left(\dfrac{1}{\sqrt{2}} \displaystyle\int_t^s \sqrt{g_k(\tau)}\, d\tau\right) & \text{if} \quad t < s \leq T, \end{cases} \tag{2.16}$$

is well defined, Lipschitz in $[0,T]$, and satisfies $\varphi_k(0) = 0$.

Since

$$h_2(\varphi_k')^2 = h_2 \frac{1}{2} g_k \varphi_k^2 \leq \frac{1}{2} h_1 \varphi_k^2 \qquad \text{in } (t,T),$$

(2.8) used with $\varphi = \varphi_k$ leads to

$$\int_t^T h_1 \varphi_k^2 ds \leq \frac{2}{t^2} \int_0^t h_2 \, ds = \frac{2}{t^2} \int_{\{u<t\}} |\nabla u|^4 \, dx = 2B_t. \tag{2.17}$$

Recall that B_t was defined in (2.13) and that we need to establish $T - t \leq C B_t^{1/2}$. By (2.15) we have

$$T - t = \int_t^T ds = \sup_{k \geq 1} \int_t^T \sqrt[4]{\frac{h_2 g_k}{h_1}} ds. \tag{2.18}$$

Using (2.17) and Cauchy-Schwarz, we have that

$$\begin{aligned}\int_t^T \sqrt[4]{\frac{h_2 g_k}{h_1}}ds &= \int_t^T \left(\sqrt{h_1}\varphi_k\right)\left(\sqrt[4]{\frac{h_2 g_k}{h_1^3}\frac{1}{\varphi_k}}\right)ds \qquad (2.19)\\ &\le (2B_t)^{1/2}\left\{\int_t^T \sqrt{\frac{h_2 g_k}{h_1^3}}\frac{1}{\varphi_k^2}ds\right\}^{1/2}\\ &\le (2B_t)^{1/2}\left\{C\int_t^T \sqrt{g_k}\frac{1}{\varphi_k^2}ds\right\}^{1/2}. \qquad (2.20)\end{aligned}$$

In the last inequality we have used our crucial estimate (2.14).

Finally we bound the integral in (2.20), using the definition (2.16) of φ_k, as follows:

$$\begin{aligned}\int_t^T \sqrt{g_k}\frac{1}{\varphi_k^2}ds &= \int_t^T \sqrt{g_k}\frac{1}{\varphi_k^2}\frac{\varphi_k'}{\frac{1}{\sqrt{2}}\sqrt{g_k}\varphi_k}ds\\ &= \sqrt{2}\int_t^T \frac{\varphi_k'}{\varphi_k^3}ds = \frac{\sqrt{2}}{2}\left[\varphi_k^{-2}(s)\right]_{s=T}^{s=t}\\ &\le \frac{\sqrt{2}}{2}\varphi_k^{-2}(t) = \frac{\sqrt{2}}{2}.\end{aligned}$$

This bound together with (2.18),(2.19), and (2.20) finish the proof in dimension $n = 4$.

Let us now turn to the

<u>*Cases $n = 3$ and 2*</u>. In these dimensions we take a simpler test function φ in (2.8) than in dimension 4. We simply consider

$$\varphi(s) = \begin{cases} s/t & \text{if} \quad s \le t\\ 1 & \text{if} \quad t < s.\end{cases}$$

With this choice of φ and since $h_1(s) \ge \int_{\Gamma_s} |A|^2 |\nabla u|\, dV_s$ —see definition (2.9)—, inequality (2.8) leads to

$$\begin{aligned}\int_t^T \int_{\Gamma_s} |A|^2 |\nabla u|\, dV_s ds &\le \int_0^T h_1(s)\varphi(s)^2 ds\\ &\le \int_0^t h_2(s)\frac{1}{t^2}ds = \frac{1}{t^2}\int_{\{u<t\}} |\nabla u|^4\, dx =: B_t.\end{aligned} \qquad (2.21)$$

This inequality and the ones that follow hold in every dimension n. It is at the end of the proof that we will need to assume $n \le 3$.

Next, we use a well known geometric inequality for the curve Γ_s ($n = 2$) or the surface Γ_s ($n = 3$). It also holds in every dimension $n \ge 2$ and it states

$$|\Gamma_s|^{\frac{n-2}{n-1}} \le C(n)\int_{\Gamma_s} |H|\, dV_s, \qquad (2.22)$$

where H is the mean curvature of Γ_s, $C(n)$ is a constant depending only on n, and s is a regular value of u. In dimension $n = 2$ this simply follows from the Gauss-Bonnet formula. For $n \geq 3$, (2.22) is stated in Theorem 28.4.1 of [5] and follows from the Michael-Simon and Allard Sobolev inequality (Theorem 2.1 of our paper). Indeed, taking $v \equiv 1$ and $m = n - 1 > 1 = p$ in (2.1), we deduce (2.22). Note that (2.22) also holds if Γ_s is not connected (with the same constant $C(n)$ as for connected manifolds).

We also use the classical isoperimetric inequality,

$$V(s) := |\{u > s\}| \leq C(n)\, |\Gamma_s|^{\frac{n}{n-1}}\,, \tag{2.23}$$

which also holds, with same constant $C(n)$, in case $\{u > s\}$ is not connected. Now, (2.22) and (2.23) lead to

$$V(s)^{\frac{n-2}{n}} \leq C(n) \int_{\Gamma_s} |H|\, dV_s \leq C(n) \left\{ \int_{\Gamma_s} |A|^2\, |\nabla u|\, dV_s \right\}^{1/2} \left\{ \int_{\Gamma_s} \frac{dV_s}{|\nabla u|} \right\}^{1/2}$$

for all regular values s, by Cauchy-Schwarz and since $|H| \leq |A|$. From this, we deduce

$$\begin{aligned} T - t = \int_t^T ds \;&\leq\; \int_t^T C(n) \left\{ \int_{\Gamma_s} |A|^2\, |\nabla u|\, dV_s \right\}^{1/2} \cdot \qquad\qquad (2.24)\\ &\qquad \cdot \left\{ V(s)^{\frac{2(2-n)}{n}} \int_{\Gamma_s} \frac{dV_s}{|\nabla u|} \right\}^{1/2} ds \\ &\leq\; C(n) \left\{ \int_t^T \int_{\Gamma_s} |A|^2\, |\nabla u|\, dV_s ds \right\}^{1/2} \cdot \\ &\qquad \cdot \left\{ \int_t^T V(s)^{\frac{2(2-n)}{n}} \int_{\Gamma_s} \frac{dV_s}{|\nabla u|} ds \right\}^{1/2} \\ &\leq\; C(n) B_t^{1/2} \left\{ \int_t^T V(s)^{\frac{2(2-n)}{n}} \int_{\Gamma_s} \frac{dV_s}{|\nabla u|} ds \right\}^{1/2}, \qquad (2.25) \end{aligned}$$

where we have used (2.21) in the last inequality.

Finally, since $V(s) = |\{u > s\}|$ is a nonincreasing function, it is differentiable almost everywhere and, by the coarea formula,

$$-V'(s) = \int_{\Gamma_s} \frac{dV_s}{|\nabla u|} \qquad \text{for a.e. } s \in (0, T).$$

In addition, for $n \le 3$, $V(s)^{\frac{4-n}{n}}$ is nonincreasing in s and thus its total variation satisfies

$$\begin{aligned} |\Omega|^{\frac{4-n}{n}} &\ge V(t)^{\frac{4-n}{n}} = \left[V(s)^{\frac{4-n}{n}} \right]_{s=T}^{s=t} \\ &\ge \int_t^T \frac{4-n}{n} V(s)^{\frac{2(2-n)}{n}} \left(-V'(s)\right) ds \\ &= \frac{4-n}{n} \int_t^T V(s)^{\frac{2(2-n)}{n}} \int_{\Gamma_s} \frac{dV_s}{|\nabla u|} ds. \end{aligned}$$

From this, (2.24), and (2.25), we conclude the desired inequality

$$T - t \le C(n) B_t^{1/2} |\Omega|^{(4-n)/(2n)}, \tag{2.26}$$

for $n \le 3$.

Note that this argument gives nothing for $n \ge 4$ since the integral in (2.25),

$$\int_t^T V(s)^{\frac{2(2-n)}{n}} \left(-V'(s)\right) ds = \int_0^{V(t)} \frac{dr}{r^{\frac{2(n-2)}{n}}}, \tag{2.27}$$

is not convergent at $s = T$ (i.e., $r = 0$) because $2(n-2)/n \ge 1$. □

3. Proof of Theorems 1.4 and 1.2

In this last section we establish Theorem 1.4 and, as a simple consequence, Theorem 1.2. They will follow easily from the following proposition. It states that, thanks to Theorem 1.1, an $L^\infty(\Omega)$ estimate for a semi-stable solution follows from having an L^∞ bound for the solution near the boundary of Ω.

Proposition 3.1. *Let f be any C^∞ function. Let $\Omega \subset \mathbb{R}^n$ be any C^∞ bounded domain. Assume that $2 \le n \le 4$.*

Let u be a classical semi-stable solution of (1.2). *Assume that*

$$u \ge c_3 \operatorname{dist}(\cdot, \partial\Omega) \quad \text{in } \Omega \tag{3.1}$$

and

$$\|u\|_{L^\infty(\Omega_\varepsilon)} \le c_4, \quad \text{where } \Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\}, \tag{3.2}$$

for some positive constants ε, c_3, and c_4.

Then,

$$\|u\|_{L^\infty(\Omega)} \le C\left(\Omega, \varepsilon, c_3, c_4, \|f\|_{L^\infty([0,c_4])}\right), \tag{3.3}$$

where $C(\cdot)$ is a constant depending only on the quantities within the parentheses.

Proof. By taking ε smaller if necessary, we may assume that $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$ is C^∞ for every $0 < \delta < \varepsilon$.

We use Theorem 1.1 with the choice

$$t = c_3 \frac{\varepsilon}{2}.$$

By (3.1), the set $\{u < t\}$ in the right hand side of our main estimate, (1.4), satisfies

$$\{u < t\} \subset \Omega_{\varepsilon/2}.$$

Thus, it suffices to bound $\|u\|_{W^{1,4}(\Omega_{\varepsilon/2})}$.

But u is a solution of $-\Delta u = f(u)$ in Ω_ε and $u = 0$ on $\partial\Omega$ (which is one part of $\partial\Omega_\varepsilon$). On the other hand, $\partial\Omega \cup \Omega_{\varepsilon/2}$ has compact closure contained in $\partial\Omega \cup \Omega_\varepsilon$, and both sets are C^∞. By (3.2), $\|u\|_{L^\infty(\Omega_\varepsilon)} \leq c_4$ and thus the right hand side of the equation satisfies $\|f(u)\|_{L^\infty(\Omega_\varepsilon)} \leq \|f\|_{L^\infty([0,c_4])}$. Hence, by interior and boundary estimates for the linear Poisson equation, we deduce a bound $\|u\|_{W^{1,4}(\Omega_{\varepsilon/2})}$ depending on the quantities in (3.3). □

The L^∞ bound (3.2) in a neighborhood of $\partial\Omega$ is known to hold for every nonlinearity f when Ω is a convex domain (in every dimension $n \geq 2$). This is proved using the moving planes method and holds for every positive solution —not only for semi-stable solutions. The precise statement is the following.

Proposition 3.2 ([18, 17, 11]). *Let f be any locally Lipschitz function and let $\Omega \subset \mathbb{R}^n$ be a C^∞ bounded domain. Let u be any positive classical solution of* (1.2).

If Ω is convex, then there exist positive constants ρ and γ depending only on the domain Ω such that for every $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) < \rho$, there exists a set I_x with the following properties:

$$|I_x| \geq \gamma \qquad \text{and} \qquad u(x) \leq u(y) \ \text{ for all } y \in I_x. \tag{3.4}$$

As a consequence,

$$\|u\|_{L^\infty(\Omega_\rho)} \leq \frac{1}{\gamma}\|u\|_{L^1(\Omega)}, \quad \text{where } \Omega_\rho = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \rho\}. \tag{3.5}$$

If Ω is not convex but we assume $n = 2$ and $f \geq 0$, then (3.5) *also holds for some constants ρ and γ depending only on Ω.*

The proof of this proposition uses the moving planes method of Gidas-Ni-Nirenberg [18]. Assume that Ω is C^∞ and convex, $n \geq 2$. For $y \in \partial\Omega$, let $\nu(y)$ be the unit outward normal to Ω at y. There exist positive constants s_0 and α depending only on the convex domain Ω such that, for every $y \in \partial\Omega$ and every $e \in \mathbb{R}^n$ with $|e| = 1$ and $e \cdot \nu(y) \geq \alpha$, we have that $u(y - se)$ is nondecreasing in $s \in [0, s_0]$. This fact follows from the reflection method applied to planes close to those tangent to Ω at $\partial\Omega$. By the convexity of Ω, the reflected caps will be contained in Ω. The previous monotonicity fact leads to (3.4), where I_x is a truncated open cone with vertex at x. If all curvatures of $\partial\Omega$ are positive this is quite simple to prove and, as mentioned in page 45 of de Figuereido-Lions-Nussbaum [17], can also be proved for convex domains with a little more of care.

In [17] it is also proved that the boundary estimate (3.5) holds also for general (nonconvex) smooth domains Ω if the nonlinearity f is subcritical in the sense that

$$f(t)t^{-\frac{n+2}{n-2}} \text{ is nonincreasing in } t \in [0, +\infty), \tag{3.6}$$

when $n \geq 3$. For these nonlinearities, we do not need to assume the convexity of Ω in our results. This result is proved with the aid of some Kelvin transforms —after which one can use the moving planes method; see [17].

When $n = 2$, Chen and Li [11] use this Kelvin transform method to establish the boundary estimate (3.5) in nonconvex domains $\Omega \subset \mathbb{R}^2$ assuming only $f \geq 0$ —as stated at the end of Proposition 3.2.

Using Propositions 3.1 and 3.2, we can now give the

Proof of Theorem 1.4. We use Proposition 3.1. We assume $f \geq 0$ —and also Ω convex in case $n \in \{3, 4\}$. Let u_k be a sequence of classical positive semi-stable solutions of (1.2) converging to u in $L^1(\Omega)$.

For $x \in \Omega$ and $v : \Omega \to \mathbb{R}$, denote

$$\delta(x) = \operatorname{dist}(x, \partial\Omega) \qquad \text{and} \qquad \|v\|_{L^1_\delta(\Omega)} = \|v\delta\|_{L^1(\Omega)}.$$

By Proposition 3.2,

$$\|u_k\|_{L^\infty(\Omega_\rho)} \leq \frac{1}{\gamma}\|u_k\|_{L^1(\Omega)} \longrightarrow \frac{1}{\gamma}\|u\|_{L^1(\Omega)}, \tag{3.7}$$

as $k \to \infty$, where ρ and γ are positive constants depending only on Ω.

Next, since $f \geq 0$, we can use a simple estimate for the linear Poisson equation $-\Delta u_k = h_k(x) := f(u_k(x)) \geq 0$ with zero Dirichlet boundary conditions. It states that

$$\frac{u_k}{\delta} \geq c\|f(u_k)\|_{L^1_\delta(\Omega)} \qquad \text{in } \Omega, \tag{3.8}$$

for some positive constant c depending only on Ω —see for instance Lemma 3.2 of [3] for a simple proof.

Multiply (1.2) (with u replaced by u_k) by the first Dirichlet eigenfunction of $-\Delta$ in Ω and integrate twice by parts. We deduce that $\|u_k\|_{L^1_\delta(\Omega)}$ and $\|f(u_k)\|_{L^1_\delta(\Omega)}$ are comparable up to multiplicative constants depending only on Ω. Multiplying (1.2) now by the solution w of

$$\begin{cases} -\Delta w &= 1 \quad \text{in } \Omega \\ \quad w &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

we deduce that also $\|u_k\|_{L^1(\Omega)}$ is comparable to the two previous quantities. Recall that $\|u_k\|_{L^1(\Omega)} \to \|u\|_{L^1(\Omega)} > 0$. Hence, the right hand side of (3.8) is bounded below by a positive constant independent of k.

As a consequence of this lower bound and of (3.7), Proposition 3.1 gives a uniform $L^\infty(\Omega)$ estimate for all solutions u_k. Letting $k \to \infty$ we deduce $u \in L^\infty(\Omega)$, as claimed in part (i) of the theorem.

To prove part (ii), we simply make more precise the constants in (3.7) and (3.8). Since we now assume $f \geq c_1 > 0$, we have that $u_k \geq c_1 w \geq c_1 c\,\delta = c_1 c \,\mathrm{dist}(\cdot, \partial\Omega)$ in Ω, where w is the solution of (3.9) and c depends only on Ω. This is estimate (3.1) of Proposition 3.1.

Finally, we multiply (1.2) (with u replaced by u_k) by the first Dirichlet eigenfunction of $-\Delta$ in Ω and integrate twice by parts. Using that $f(s) \geq \mu s - c_2$ for all s and that $\mu > \lambda_1$, we obtain a control $\|u_k\|_{L^1_\delta(\Omega)} \leq \overline{C} = \overline{C}(\Omega, \mu, c_2)$ and thus also for $\|u_k\|_{L^1(\Omega)}$ as mentioned before. Now, this estimate combined with (3.7) give an estimate as (3.2). Estimate (3.3) in Proposition 3.1 gives the desired conclusion (1.8) of Theorem 1.4. □

Theorem 1.2 on the boundedness of the extremal solution u^* follows easily from Theorem 1.4.

Proof of Theorem 1.2. We extend g in a C^1 manner to all of $\mathbb{R}$ with g nondecreasing and $g \geq g(0)/2$ in $\mathbb{R}$. Recall the the extremal (weak) solution u^* is the increasing L^1 limit, as $\lambda \uparrow \lambda^*$, of the minimal solutions u_λ of (1.5_λ). In addition, for $\lambda < \lambda^*$, u_λ is a C^2 semi-stable solution of (1.5_λ) —see Remark 1.3 in the introduction.

If g is C^∞ we simply apply part (ii) of Theorem 1.4 with $f = \lambda g$ for $\lambda^*/2 < \lambda < \lambda^*$. Using that g satisfies (1.6), we can verify (1.7) and obtain estimates for $\|u_\lambda\|_{L^\infty(\Omega)}$ which are uniform in λ. Letting $\lambda \uparrow \lambda^*$ we conclude that $u^* \in L^\infty(\Omega)$.

In case that $g \in C^1$ is not C^∞, let ρ_k be a C^∞ mollifier with support in $(0, 1/k)$, of the form $\rho_k(\beta) = k\rho(k\beta)$. We replace g by

$$g_k(s) = \int_{s-1/k}^{s} g(\tau)\rho_k(s-\tau)\, d\tau = \int_0^1 g(s - \beta/k)\rho(\beta)\, d\beta.$$

For all k, we have that $g_k \leq g_{k+1} \leq g$ in $\mathbb{R}$, g_k is C^∞, and (as g) nondecreasing. In addition, g_k satisfies all conditions in (1.6). Since $g(u^*) \geq g_k(u^*)$, u^* is a weak supersolution for problem (1.5_{λ^*}) with g replaced by g_k. By the monotone iteration procedure, it follows that the extremal parameter for g_k, λ_k^*, satisfies $\lambda^* \leq \lambda_k^*$. Hence $u^k_{\lambda^*-1/k}$, the solution for problem (1.5_{λ^*}) with g replaced by g_k and with $\lambda = \lambda^* - 1/k$ is classical. Thus, we can apply Theorem 1.4 with $f = \lambda g_k$ and $\lambda = \lambda^* - 1/k$ to obtain an $L^\infty(\Omega)$ bound for $u^k_{\lambda^*-1/k}$ independent of k. Note that $u^k_{\lambda^*-1/k} \leq u^k_{\lambda^*-1/(k+1)}$ and that, since $g_k \leq g_{k+1} \leq g$, $u^k_{\lambda^*-1/(k+1)} \leq u^{k+1}_{\lambda^*-1/(k+1)} \leq u_{\lambda^*} = u^*$. Thus, $u^k_{\lambda^*-1/k}$ increases in $L^1(\Omega)$ towards a solution of (1.5_{λ^*}) smaller or equal than u^*, and hence identically u^*. From the $L^\infty(\Omega)$ bound for $u^k_{\lambda^*-1/k}$ independent of k, we conclude $u^* \in L^\infty(\Omega)$. □

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